Proof of the twin-prime conjecture

Preliminary

This proof relies on my proof of the Goldbach conjecture.
The following elements and notational conventions of this proof are relevant for the proof of the twin-prime conjecture:

- The ordered pair (P_a , P_b) is referred to as a G-pair (Goldbach pair), where $P_a \leq P_b$
- Sets are denoted by a capital S, with a subscript.
- Prime numbers are denoted by the capital letter P with a subscript.
- What is meant by a "#" depends on the context:
  - If it relates to a Set, #(S_x) should be read as "the number of elements in Set x".
  - If it refers to a number that is an element of an ordered set, #(n) should be read as "the n^{th} element of the set".
- The set S_d [d for: divisors] consists of all prime numbers P_d for which holds:

  (P_d) ≤ √(2n); (P_d >2)

  Given the sieve of Eratosthenes, these are all prime numbers that can be the smallest divisor of a number ≤ 2n. Because 2n is even by definition, prime divisor 2 does not signify any difference between different values of 2n. Therefore, it is not counted as an element of the set S_d. The set is ordered, i.e. each element has a cardinal number, starting at P_d (#1) = 3.
- The set S_a consists of all prime numbers P_a that can be the smallest member (P_a) of a G-pair, given 2n. $P_a \leq n$; P_a >2. The set is ordered, i.e. each element has a cardinal number, starting with P_a (#1) = 3
- P_{az} is the upper limit for the smallest element (P_a) of a G-pair. It is not possible to create a construction where all elements $P_a \leq P_{az}$ are the smallest element of a G-pair.
- P_{az} can be (over-) estimated in the following way:

  Fraction (#S_d) = $\prod_{k=1}^{k=\#Sd} [P_k - 2]/[P_k - 1]$

  MI (#S_d) = 1/Fraction (#S_d)

  #P_{az} < #S_d* MI (#S_d)

In the example given (#S_d = 45) we calculate that
#S_d = 45; MI (#45) = 7,284.

$39601 \leq 2n < 44521$.

$\#S_a = 2240$;

$\#P_{az} \leq (45 \times 7{,}284 =) 328$

$P(\#328) = 2207$

→ the smallest element P_a of at least one G-pair of 2n is 2207 or smaller.

It also follows that (*see: Goldbach proof, bonus 3*),

For any G-pair = $((n - Z),(n + Z))$

$\#P_x = \#S_a - \#P_{az}$

$Z < n - P_x$

In the example given: $\#P_x = 2240 - 328 = 1912$;

$P(\#1912) = 16493$

$Z < (19801 - 16493 =) 3308$

1 Transformation of the question

All twin-primes are ordered pairs of prime numbers (P_k , P_{k+1})

for which holds that $P_k = (6m - 1)$ and $P_{k+1} = (6m+1))$.

We do not (primarily) look at the values for P_k, but only at possible values for m.

2 Negative Conditions

If	6m +/- 1 = a	a is prime
Then	$6 * (na + m) \mp 1$	| a
	$6 * (na - m) \pm 1$	| a

Example: m = 3;

a = (6 * 3) – 1 = 17

→	(6 * (0 * 17) + 3)) – 1 = (3 * 6)- 1	= 1 * 17
	(6 * (1 * 17) + 3)) – 1 = (20 * 6) - 1	= 7 * 17
	(6 * (2 * 17) + 3)) – 1 = (37 * 6) -1	= 13 * 17
	(6 * (3 * 17) + 3) – 1 = (54 * 6) -1	= 19 * 17
	….	
	(6 * (1* 17) - 3) + 1 = (14 * 6) - 1	= 5 * 17

(6 * (2 * 17) - 3) + 1 = (31 * 6) - 1 = 11 * 17

(6 * (3* 17) - 3) + 1 = (38 * 6) - 1 = 17 * 17

In this example it can be seen that

if m ≡ {3,14} mod 17,

then 6m-/+ 1 |17;

→ ((6m-1),(6m+1)) is not a twin-prime.

In this example, "m ≡ {3,14} mod 17" constitutes a Negative Condition [from here: NC] for m.
An NC means that we set the condition that m is *not* (3 mod 17) or (14 mod 17), because in that case we know for sure that 6m +/- 1 is not a twin-prime.

notational note: we use the character ≠ for NC's.
So the statement of the NC becomes:
m ≠ {3,14} mod 17

Each NC consists of two complementary negative sub-conditions:
[x mod P] mod P and [(P - x) mod P]
[with the exception of P = 2 , because the only valid value for x is 1, and therefore x = P – x.]
Each prime number determines an NC for m.

The subscript NC_k indicates that all NC's that relate to primes up to and including P_k apply to m.

3 upper limit for m; reformulation of the question

The upper limit for m (m_g) is determined by NC_k.
The first value of m that satisfies all NC_k but *does not* necessarily lead to a twin-prime, is the first value of m where (6m +/- 1) is non-prime, with a the smallest divisor a prime number > P_k.
The first non-prime number with Lowest Divisor > P_k is $(P_{k+1})^2$.
$((P_{k+1})^2 - 1) \mid 6$ [for all primes it holds that $(P^2 – 1)$ is divisible by 6]
→ $m_g = ((P_{k+1})^2 – 1)/6$.

If m, for $m < m_g$, satisfies NC_k then m defines the twin pair ((6m-1), (6m+1)).
This is because we know for sure that (6m +/-1) is not divisible by a prime number ≤ P_k, and it can not be a non-prime divisible by a smallest devisor > P_k

Example:

Voor $P_k \leq 13$ the following NC's apply:

$m \neq \{1,4\} \bmod 5$

$m \neq \{1,6\} \bmod 7$

$m \neq \{2,9\} \bmod 11$

$m \neq \{2,11\} \bmod 13$

$P_{k+1} = 17$

$m_g = (17^2 - 1)/6 = 48$

From this it follows that any value of m < 48 that satisfies the 4 NC's defines a twin-prime.

m = {3, 5, 7, 10, 12, 17, 18, 23, 25, 30, 32, 33, 38, 40, 45, 47}

Note: by satisfying the condition $m \neq 1$, the twin (5,7) to be absent; similarly, $m \neq 2$ causes the twin (11,13) to be absent.

If the number of twin-primes is finite, there exists a value for P_k such that there is no m that satisfies all NC's. That is, if P_k is the part of the highest twin-prime.

To be proven: For any k, there exists at least one $m < (P_{k+1}^2 - 1)/6$, for which m satisfies all NC_k.

4 Relationship with the Goldbach conjecture

The Goldbach conjecture can be also formulated in terms of a number (Z) that must satisfy a number of NC's. I will demonstrate this for 2n = 82:

For Goldbach, the following possible procedure applies:

1. Rewrite 2n as n + n

   82 = 41 + 41

   This is a G-pair if

   (41 + Z) is prime and

   (41 – Z) is prime.

2. Describe n as a set of dividends and divisors, with all primes up to k as divisors, k being the largest possible Smallest Divisor of 2n:

{

41 ≡ 1 mod 2

41 ≡ 2 mod 3

41 ≡ 1 mod 5

41 ≡ 6 Mod 7

}

3. Based on this, the following sub-NC's apply to Z:

Z ≠ 1 mod 2 (because then: 41 –/+ Z divisible by 2)

Z ≠ (1,2) mod 3 (because then: 41 +/– Z divisible by 3)

Z ≠ (1,4) mod 5 (because then: 41 –/+ Z divisible by 5)

Z ≠ (1,6) mod 7 (because then: 41 +/- Z divisible by 7)

Z < 41

Goldbach's conjecture states that for every value of n, at least 1 valid value of Z must exist.
Goldbach's conjecture is presumed to be proven (see above)
In this example, the set of numbers that satisfy all NC's: Z = {0, 12, 18, 30}

The *similarity* between the Goldbach conjecture and the twin-prime conjecture is that both can be formulated in terms of NC's that some number must satisfy.
The *difference* is that in Goldbach's conjecture, the divisors 2 and 3 also set NC's for Z, while these divisors do not set NC's for m in the twin-prime conjecture.

5 solution for Goldbach are solutions for twin-primes

The NCs for m taken together create series of numbers.
Together, these are all numbers that *do not* satisfy one or more NC's.
Each NC added doubles the number of series:

NC {1, 4} mod 5

+NC {1, 6} mod 7

→ 4 series: {1,6,29,34} mod 35

+NC {2, 9} mod 11

→ 8 series {64, 134, 141, 174, 211, 244, 251, 321} mod 385

+NC {2,11} mod 13

→ 16 series: {141, 596, 834, 1289, 1406,1714 .... 4409, 4816} mod 5005

We use the elements from these series as input values to determine a G-pair:

Example 1:

Starting from 141, in the series [m ≡ 141 mod 5005]:

We use 141 as n in a G-pair; 2n = 282.

The number 282 can be written as the sum of 2 primes:

282 = (141 - Z) + (141+ Z)

For a valid G-pair, Z must satisfy NC_{13}:

i.e.: Z ≠ {1,4} mod 5, Z ≠ {1,6} mod 7 etc. up to Z ≠ {2,11} mod 13; [because $282 < 17^2$, i.e. the smallest divisor of Z can never be over 13]

Z < 141

$P_{k+1} = 17$

$m_g < (17^2 - 1) / 6$

→ $m_g < 48$

For a valid twin-prime, $Z < m_g$:

Every Z that meets these NC's is a solution for a G-pair.

Every Z that meets these NC's, ((6Z -1),(6Z+1)) is a twin-prime.

After all, Z and m must comply with exactly the same NC_{13}.

Solutions are: Z = {**10,32,38,40**}

Note 1: There are also solutions Z > 48, i.e. {52, 58, 70, 82, 98, 100, 126, 128, 130} – but these do not guarantee a twin-prime, because $Z > m_g$

Note 2: These are not *all* the solutions for m. When determining Z for 141, possible solutions have also been lost due to the fact that (141 +/- Z) is divisible by 2 or by 3. As a result, for example, all odd values for Z are out in advance.

The Z-values that do not show up at 141 are the values {3, 5, 7, 12, 17, 23, 25, 30, 33, 45}. These are all valid solutions for m.

Note 3: Apart from the values for Z that are eliminated due to $Z > m_g$, **all** solutions for Z are also valid solutions for m

Note 4: It doesn't matter how high the value of 2n is in the G-pair – it's about the value of Z. If $Z < m_g$, this always gives a valid value for m. So even if we assume n = 4816, all Z-values (Z <48) are valid solutions for m.

Example 2:

Following the example from the evidence of the Goldbach conjecture:

$\#S_d = 45$; $P(45) = 199$. [I.e. that all primes up to 199 were used to determine NC's]

In the Goldbach's 'bonus' section, it shown that there must always be a Z for which the following applies:

$\#P_x = \#S_a - \#P_{az}$

$Z < n - P_x$

In the example:

$\#S_d = 45$; $P_{45} = 199$;

$39601 \leq 2n < 44521$;

$P_{46} = 211$; $m_g = ((211^2 - 1)/6) =$ **7420**;

$\#S_a = 2240$;

$\#P_{az} = 328$;

$\#P_g = 2240 - 328 = 1912$;

$P_{1912} = 16493$;

$Z < ((39601)/2 - 16493 =) 3308$;

3308 $< m_g$

Now all that has to be proven is, that $Z < m_g$

If this is always the case, there is at least 1 twin-prime for every $\#S_d$

6 Full Induction

Z is determined by

1) by the mean prime-gap between the primes between n and n – Z;

2) the number of prime numbers $\#P_{az}$.

For n $\rightarrow\infty$, the average prime gap increases infinitely slowly. [follows from the Prime Number Theorem] This change should be neglected at a sufficiently large n.

Doubling $\#S_d$ leads to doubling $\#P_{az}$ [at sufficiently large n]; This also follows from the PNT.

It follows that, in the limit for n $\rightarrow\infty$, a doubling of $\#S_d$ leads to a doubling of Z. (a)

m_g is determined by $(P_{k+1}^2 - 1)/6$,

A doubling of $\#S_d$ [for sufficiently large n] leads approximately to a doubling of P_k, and therefore also to an approximate doubling of P_{k+1}.

$\rightarrow$ Doubling of $\#S_d$ leads to a quadrupling of m_g. (b)

a Doubling of $\#S_d$ leads to a doubling of Z

b Doubling of $\#S_d$ leads to a quadrupling of m_g

$\rightarrow$ Z and m_g diverge

At $\#S_d = 45$ $Z < m_g$

Z and m_g diverge

$\rightarrow$ $Z < m_g$ for $\#S_d > 45$

It follows that, for $\#S_d > 45$, there is always at least 1 twin-prime.

This is because on the one hand, with increasing n, the lower solutions are excluded from Z. So where in the smaller example (n = 141) there was a valid solution for Z = 10, that solution is no longer valid if the NC for 101 is included, as it is in the second example.

On the other hand, with increasing n, increasingly higher solutions are permitted, since m_g increases.

Taken together, it is always possible to choose an n that excludes all previous solutions for Z, and still yields a valid solution, given Goldbach's conjecture. This excludes the possibility that there is only a limited number of values for m.

It follows that there are an infinite number of twin-primes.